Answer Key

Final 09/010 (fall)

I- **(35pts - 5pts each)** Calculate the following integrals:

1) $\int \frac{3^x + 4^x}{5^x} dx = \int \frac{3^x}{5^x} dx + \int \frac{4^x}{5^x} dx = \int \left(\frac{3}{5}\right)^x dx + \int \left(\frac{4}{5}\right)^x dx$

$$= \boxed{\left(\frac{3}{5}\right)^x \div \ln\left(\frac{3}{5}\right) + \left(\frac{4}{5}\right)^x \div \ln\left(\frac{4}{5}\right) + k}$$

$$= \boxed{\frac{\left(\frac{3}{5}\right)^x}{\left(\ln \frac{3}{5}\right)} + \frac{\left(\frac{4}{5}\right)^x}{\ln\left(\frac{4}{5}\right)} + k}$$

2) $\int \frac{dx}{(4 - x^2)^{3/2}}$

Let $x = 2\sin\alpha$

$dx = 2\cos\alpha \, d\alpha$

$4 - x^2 = 4 - 4\sin^2\alpha = 4(1 - \sin^2\alpha) = 4\cos^2\alpha$

$\sqrt{4 - x^2} = 2\cos\alpha$

$$I = \int \frac{2\cos\alpha \, d\alpha}{(2\cos\alpha)^3} = \int \frac{2\cos\alpha \, d\alpha}{8\cos^3\alpha} = \frac{1}{4}\int \frac{d\alpha}{\cos^2\alpha}$$

$$= \frac{1}{4}\int \sec^2\alpha \, d\alpha = \frac{1}{4}\left[\tan\alpha\right] + k$$

$$= \frac{1}{4}\left[\frac{\sin\alpha}{\cos\alpha}\right] + k = \frac{1}{4}\left[\frac{\frac{x}{\cancel{2}}}{\frac{\sqrt{4-x^2}}{\cancel{2}}}\right] + k$$

$$= \boxed{\frac{1}{4}\left[\frac{x}{\sqrt{4-x^2}}\right] + k}$$

3) $\int \frac{\ln x}{x^5}\,dx$

Let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$v = \frac{-1}{4x^4} \Leftarrow dv = \frac{dx}{x^5}$

$$I = \left(\frac{-1}{4x^4}\right)(\ln x) - \int \frac{-dx}{4x^5}$$

$$\boxed{I = \left(\frac{-1}{4x^4}\right)(\ln x) - \frac{1}{16x^4} + k}$$

4) $\int x\cos x \sin x\,dx$ $= \int x \frac{\sin 2x}{2}\,dx = \frac{1}{2}\int x \sin 2x\,dx$

| | |
|---|---|
| $x$ | $\sin 2x$ |
| $1$ $(+)$ | $-\frac{\cos 2x}{2}$ |
| $0$ $(-)$ | $-\frac{\sin 2x}{4}$ |

$$I = \frac{1}{2}\left[-\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x\right] + k$$

$$\boxed{I = -\frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x + k}$$

5) $\int(\tan x)^5\,dx = \int \tan^2 x \cdot \tan^3 x\,dx$

$= \int (\sec^2 x - 1)\tan^3 x\,dx$

$= \int \sec^2 x \tan^3 x\,dx - \int \tan^3 x\,dx$

Let $u = \tan x$
$du = \sec^2 x$

$= \dfrac{\tan^4 x}{4} - \int \tan^2 x \cdot \tan x\,dx$

$= \dfrac{\tan^4 x}{4} - \int (\sec^2 x - 1)\tan x\,dx$

Let $u = \tan x$
$du = \sec^2 x\,dx$

$I = \dfrac{\tan^4 x}{4} - \dfrac{\tan^2 x}{2} + \int \tan x\,dx$

$\boxed{= \dfrac{\tan^4 x}{4} - \dfrac{\tan^2 x}{2} - \ln|\cos x| + k}$

6) $\int \dfrac{x^5+1}{(x^3)(x+3)}\,dx$

$$\begin{array}{r|l} & x-3 \\ \hline x^4+3x^3 & x^5+1 \\ & -x^5-3x^4 \\ \hline & -3x^4+1 \\ & +3x^4+9x^3 \\ \hline & 9x^3+1 \end{array}$$

$\dfrac{9x^3+1}{x^3(x+3)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x^3} + \dfrac{D}{x+3}$

$\begin{cases} C = \frac{1}{3} \\ D = \frac{242}{27} \\ A = \frac{1}{27} \end{cases}$ $B = \dfrac{-1}{9}$

$I = \int x - 3 + \dfrac{9x^3+1}{x^3(x+3)}\,dx$

$I = \int \left(x - 3 + \dfrac{\frac{1}{27}}{x} + \dfrac{\frac{-1}{9}}{x^2} + \dfrac{\frac{1}{3}}{x^3} + \dfrac{242/27}{x+3}\right)dx$

$\boxed{I = \dfrac{x^2}{2} - 3x + \dfrac{1}{27}\ln|x| + \dfrac{1}{9x} - \dfrac{1}{6x^2} + \dfrac{242}{27}\ln|x+3| + k}$

7) $\int \frac{3}{4+x^{1/3}}\,dx$

Let $u = x^{\frac{1}{3}} \Rightarrow du = \frac{dx}{3x^{2/3}}$

$dx = 3x^{2/3}\,du$

$dx = 3u^2\,du$

$$\begin{array}{r} u-4 \\ 4+u\,\overline{)\,u^2} \\ \underline{-u^2-4u} \\ -4u \\ \underline{+4u+16} \\ 16 \end{array}$$

$$I = \int \frac{3 \cdot 3u^2\,du}{4+u} = 9\int \frac{u^2\,du}{4+u}$$

$$= 9\int \left(u-4+\frac{16}{u+4}\right) du$$

$$= 9\left[\frac{u^2}{2} - 4u + 16\ln|u+4|\right] = \boxed{9\left[\frac{x^{2/3}}{2} - 4x^{\frac{1}{3}} + 16\ln\left(x^{\frac{1}{3}}+4\right)\right] + k}$$

II- **(10pts - 5pts each)** Find the following derivatives:

1) $y = x\sec^{-1}x - \sqrt{1-x^2} + 2\sqrt{x-1}\,\sec^{-1}\sqrt{x}$

$$y' = (1)(\sec^{-1}x) + \left(\frac{1}{\cancel{x}\sqrt{x^2-1}}\right)(\cancel{x}) + \frac{x}{\sqrt{1-x^2}} + \left(\frac{1}{\sqrt{x-1}}\right)(\sec^{-1}\sqrt{x})$$

$$+ \left(\frac{\frac{1}{\cancel{2}\sqrt{x}}}{\sqrt{x}\cdot\cancel{\sqrt{x-1}}}\right)\left(\cancel{2\sqrt{x-1}}\right)$$

$$\boxed{y' = \sec^{-1}x + \frac{1}{\sqrt{x^2-1}} + \frac{x}{\sqrt{1-x^2}} + \frac{\sec^{-1}\sqrt{x}}{\sqrt{x-1}} + \frac{1}{x}}$$

2) $$y=\left(\frac{2x4^x}{\sqrt{x^2+1}}\right)^3(\tan x)^2\frac{(5x+4)^3}{(2x-1)^2}$$

$$\ln y = 3\left[\ln 2+\ln x + x\ln 4-\frac{1}{2}\ln(x^2+1)\right]+2\ln(\tan x)+3\ln(5x+4)-2\ln(2x-1)$$

$$\frac{y'}{y}=3\left[\frac{1}{x}+\ln 4-\frac{x}{x^2+1}\right]+2\left(\frac{\sec^2 x}{\tan x}\right)+\frac{15}{5x+4}-\frac{4}{2x-1}$$

$$\boxed{y'=\left[\left(\frac{2x4^x}{\sqrt{x^2+1}}\right)^3(\tan x)^2\frac{(5x+4)^3}{(2x-1)^2}\right]\left\{3\left[\frac{1}{x}+\ln 4-\frac{x}{x^2+1}\right]+2\left(\frac{\sec^2 x}{\tan x}\right)+\frac{15}{5x+4}-\frac{4}{2x-1}\right\}}$$

III- **(10pts - 5pts each)** Solve the following:

1) Using the shell method, find the volume of the solid generated by revolving about the y-axis, the region bounded by: $y=x^2$ & $y=\sqrt{x}$.

$y_1=y_2 \Rightarrow x^2=\sqrt{x}$

$x=0,\ x=1$

$$V=\left|\int_0^1 2\pi x\left(x^2-\sqrt{x}\right)dx\right|=\left|2\pi\int_0^1\left(x^3-x^{3/2}\right)dx\right|$$

$$V=\left|2\pi\left[\frac{x^4}{4}-\frac{x^{5/2}}{5/2}\right]_0^1\right|=\left|2\pi\left(\frac{1}{4}-\frac{2}{5}\right)\right|=\boxed{\frac{6\pi}{20}}$$

$$=\boxed{\frac{3\pi}{10}}$$

2) Find the length of the curve $y = \ln(\sec x)$ from $x = 0 \, to \, x = \frac{\pi}{4}$

$$y = \ln(\sec x) \Rightarrow y' = \frac{\sec x \tan x}{\sec x} = \tan x$$

$$y'^2 + 1 = \tan^2 x + 1 = \sec^2 x$$

$$\sqrt{1+(y')^2} = \sqrt{\sec^2 x} = \sec x$$

$$L = \int_0^{\pi/4} \sqrt{1+(y')^2}\, dx = \int_0^{\pi/4} \sec x \, dx = \ln\left|\sec x + \tan x\right| \Big]_0^{\frac{\pi}{4}}$$

$$L = \ln(\sqrt{2}+1) - \ln(1) = \boxed{\ln(\sqrt{2}+1)}$$

IV_ **(10pts - 5pts each)** Solve for $x$:

1) $(\ln x)^3 - 5(\ln x)^2 + 6\ln x = 0$

Let $u = \ln x \Rightarrow$

$$u^3 - 5u^2 + 6u = 0$$

$$u(u^2 - 5u + 6) = 0 \Rightarrow u(u-2)(u-3) = 0$$

$u = 0$, $\ln x = 0$, $\boxed{x = 1}$

$u = 2$, $\ln x = 2$, $\boxed{x = e^2}$

$u = 3$, $\ln x = 3$, $\boxed{x = e^3}$

2) $\dfrac{e^{5x+4}}{e^{3x-2}} = e^{2x+4}$

$$e^{5x+4} = \left(e^{2x+4}\right)\left(e^{3x-2}\right)$$

$$e^{5x+4} = e^{5x+2}$$

$$\Rightarrow \cancel{5}x+4 = \cancel{5}x+2$$

$4 \neq 2$ impossible no values of $x$ can be found.

V- **(10pts - 5pts each)** Test the following integrals for convergence. If they converge, find their limits:

a. $\int_0^\infty x(1+x)^{-5}\,dx = \int_0^\infty \dfrac{x}{(1+x)^5}\,dx$

$$\lim_{n\to\infty} \frac{\frac{x}{(1+x)^5}}{\frac{1}{x^4}}$$

By H.R.

$$= \lim_{n\to\infty} \frac{x^5}{(1+x)^5} = 1$$

Both Conv. or both Div.

$\Rightarrow \int_0 \frac{1}{x^4}\,dx$ ~~Conv~~ Div.

$\Rightarrow \int_0^\infty x(1+x)^{-5}\,dx$ ~~Conv.~~ Div.

$1+x > x$

$\frac{1}{1+x} < \frac{1}{x}$

$\frac{1}{(1+x)^5} < \frac{1}{x^5}$

$\frac{x}{(1+x)^5} < \frac{x}{x^5} = \frac{1}{x^4}$

$\int_0^\infty \frac{x\,dx}{(1+x)^5} < \int_0^\infty \frac{dx}{x^4} = \frac{-1}{3x^3}\Big|_0^\infty$

$= 0 - (-\infty) = +\infty$

b. $\int_1^\infty \frac{1}{x\left(\sqrt{\ln x}+\ln^2 x\right)}dx$

Let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$$I = \int_0^\infty \frac{du}{\sqrt{u}+u^2}$$

$\cancel{u^2+\sqrt{u} > u^2}$

$\cancel{\frac{1}{u^2+\sqrt{u}} < \frac{1}{u^2}}$

$\cancel{\int \frac{du}{u^2+\sqrt{u}} < \int \frac{du}{u^2}} = -\frac{1}{u}\Big]_0^\infty$

$= 0 - [-\infty]$

$= \infty$

$\Rightarrow \boxed{I \ \cancel{\text{Conv}}}$ Div by limit

$$\lim_{u\to\infty} \frac{\frac{1}{u^2}}{\frac{1}{\sqrt{u}+u^2}} = \lim_{u\to\infty} \frac{\sqrt{u}+u^2}{u^2} = \lim_{u\to\infty}\left(\frac{1}{u^{\frac{3}{2}}}+1\right) = 1$$

~~Conv~~ diverges.

VI- **(5pts)** Solve for $x$ when $\sin\left(\tan^{-1}\frac{x}{\sqrt{x^2+1}}\right) = \frac{2}{6}$

Let $y = \tan^{-1}\frac{x}{\sqrt{x^2+1}} \Rightarrow \tan y = \frac{x}{\sqrt{x^2+1}}$

$\sin y = \frac{2}{6} = \frac{1}{3}$

$\tan y = \frac{\sin y}{\cos y} = \frac{\sin y}{\sqrt{1-\sin^2 y}}$

$\frac{x}{\sqrt{x^2+1}} = \frac{\frac{2}{6}}{\sqrt{1-\frac{4}{36}}} = \frac{\frac{2}{6}}{\sqrt{\frac{32}{36}}} = \frac{2}{6} \times \frac{6}{\sqrt{32}} = \frac{1}{\sqrt{8}}$

$\frac{x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{8}} \Rightarrow \frac{x^2}{x^2+1} = \frac{1}{8}$

$8x^2 = x^2+1 \Rightarrow 7x^2 = 1 \Rightarrow x^2 = \frac{1}{7}$

$\boxed{x = \frac{1}{\sqrt{7}}}$ $x = -\frac{1}{\sqrt{7}}$ rejected $\boxed{x = \pm\frac{1}{\sqrt{7}}}$

note:

$$\int_1^e \ln u\,du = u\ln u - u\Big]_1^e = (e\ln e - e) - \underbrace{(0-1)}_{=1}$$

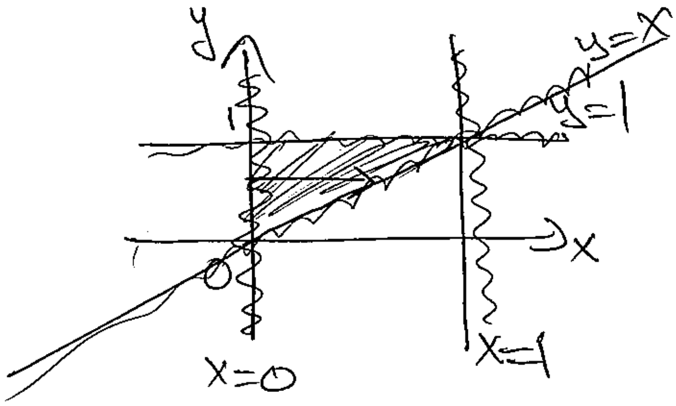
VII- **(10pts - 5pts each)** Evaluate the following integrals:

a) $\int_1^e \int_1^e \int_1^e (\ln x \ln y \ln z)\,dz\,dy\,dx$

$$= \int_1^e\int_1^e \ln x \ln y \underbrace{\left(\int_1^e \ln z\,dz\right)}_{1} dy\,dx$$

$$= \int_1^e\int_1^e \ln x \ln y\,dy\,dx = \int_1^e \ln x \underbrace{\left(\int_1^e \ln y\,dy\right)}_{1} dx = \int_1^e \ln x\,dx = \boxed{1}$$

b) $\int_0^1 \int_x^1 e^{\frac{x}{y}}\,dy\,dx$




$$= \int_0^1\int_0^y e^{\frac{x}{y}}\,dx\,dy = \int_0^1 \frac{e^{x/y}}{\frac{1}{y}}\Big]_0^y dy = \int_0^1 y e^{x/y}\Big]_0^y dy$$

$$= \int_0^1 (ye - y)\,dy = (e-1)\int_0^1 y\,dy = (e-1)\frac{y^2}{2}\Big]_0^1$$

$$= \boxed{\frac{(e-1)}{2}}$$

VIII- **(10pts - 5pts each)** Solve the following problems:

a. Find the acute angle between the 2 vectors:

$$\vec{u}=2\vec{i}-\vec{j} \quad \& \vec{v}=4\vec{i}+\frac{3}{2}\vec{j}$$

$$\alpha = \cos^{-1}\left(\frac{(2)(4)+(-1)\left(\frac{3}{2}\right)}{\sqrt{2^2+(-1)^2}\cdot\sqrt{4^2+\left(\frac{3}{2}\right)^2}}\right) = \cos^{-1}\left(\frac{\frac{13}{2}}{\sqrt{5}\cdot\frac{\sqrt{73}}{2}}\right)$$

$$\boxed{\alpha = \cos^{-1}\left(\frac{13}{\sqrt{205}}\right)} \qquad \alpha = \cos^{-1}\left(\frac{13}{\sqrt{365}}\right)$$

b. Find the unit vector(s) that are parallel and normal to the vector $\vec{v}=\vec{i}-4\vec{j}$

$$\vec{U}_{\parallel} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{i}-4\vec{j}}{\sqrt{1+(-4)^2}} = \boxed{\frac{1}{\sqrt{17}}\vec{i}-\frac{4}{\sqrt{17}}\vec{j}}$$

$$\underline{\underline{OR}} \quad \boxed{\frac{-1}{\sqrt{17}}\vec{i}+\frac{4}{\sqrt{17}}\vec{j}}$$

$$\vec{U}_{\perp} = \boxed{\frac{4}{\sqrt{17}}\vec{i}+\frac{1}{\sqrt{17}}\vec{j}}$$

$$\underline{\underline{OR}} \quad \boxed{-\frac{4}{\sqrt{17}}\vec{i}-\frac{1}{\sqrt{17}}\vec{j}}$$